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A new quantum mechanical derivation of the photocounting distribution

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Abstract. We present a new direct derivation of the photocounting distribution. Our quantum mechanical formulation is valid to all orders in perturbation theory.

1. Introduction

Some of the theories of photoelectric detection that have been proposed (see for example Glauber 1964, Kelley and Kleiner 1964, Lehmberg 1968, Rocca 1973, Arnedo and Rocca 1974, Klauder and Sudarshan 1968 (see § 8), Rousseau 1975) are valid only to first order in perturbation theory.

In these theories meaningless results appear for large times. For example, the probability $P^{(1)}(t)$ for a single atom to be ionized by an incoming mono-mode light grows linearly with time, instead of being bounded by unity; moreover in the case of the n-photon field, the probabilities become negative for large sampling time. These unsavoury features disappear when multiple transitions between the excited states and the ground state are taken into account. In a complete quantum mechanical treatment the field is attenuated as a result of the detection process itself (Mollow 1968, Scully and Lamb 1969). The approach we present here is more direct, because we have only to study the simple case of a photodetector inserted in a one-photon field (§ 2). This result together with a previously published result of the independency of atoms (Rocca 1973, Arnedo and Rocca 1974) provides the counting distribution for a detector immersed in: (i) an n-photon field; (ii) an arbitrary field (described by a density matrix on the Fock basis or on the coherent states basis (§ 3)). We show that the photocounting distribution can be expressed as a compound Poisson distribution. The characteristic parameter of this distribution is proportional to the reduced intensity because of the depletion phenomena due to the photodetector.

2. Photodetector in a one-photon field

2.1. Single atom in a one-photon field

Consider first a single atom in a one-photon field. It is shown in the appendix that the probability for the atom to be ionized in the time interval $(0, t)$ is

$$P^{(1)}(t)=1-\left\langle\exp\left(-s\int_0^t I(\theta)\,\mathrm{d}\theta\right)\right\rangle \qquad (1)$$

where $I(\theta)$ is the intensity of the field and the angular bracket denotes an ensemble average. According to first-order perturbation theory, the transition probability is (Glauber 1964):

$$P^{(1)}(t) = s\langle I\rangle t, \tag{2}$$

an expression which agrees with equation (1) for small times. Equation (2) is unacceptable for large times since the probability is unbounded whereas equation (1) is valid at all times.

For an n-photon field the transition probability is (see the appendix):

$$P^{(1)}(t) = 1 - \mathrm{e}^{-stn}. \tag{3}$$

This expression is very similar to the one in equation (1), because strong fields can be considered as n-photon fields with $I(\theta) \sim n$. This property of strong fields follows from the photon distribution $\rho_{nn} = \mathrm{e}^{-I} I^n/n!$ which gives $\sigma_n^2 = \bar{n}$; therefore, for strong fields, $\sigma_n/\sqrt{\bar{n}}$ tends to zero as $\bar{n}$ tends to infinity. Semi-classical and quantum theories thus give the same transition probabilities (Oliver 1971).

2.2. *An N-atom detector*

Next, consider an N-atom detector. Each atom can be ionized by the incident electromagnetic field with the probabilities given above.

The Schrödinger equation for the whole system (atoms + field) can be decomposed into N first-order differential equations. The time derivative of the probability amplitude $b_{K_1 \ldots K_p}(g_1 \ldots a_1 \ldots a_p \ldots g_N; t)$ that p photons of modes $K_1 \ldots K_p$ are absorbed and p atoms are in states $a_1 \ldots a_p$, analogous to the $b(g, t)$ used in the appendix, is then a linear combination of $b_{K_1 \ldots K_{p-1}}$ and $b_{K_1 \ldots K_{p+1}}$. In general, this system of equations cannot be solved in closed form. For the one-photon field, however, we have only two equations similar to equations (A.3) and (A.4) in the appendix.

The probability that N atoms remain in the ground state at time t is then

$$|b(g_1 \ldots g_N, t)|^2 = \mathrm{e}^{-Nst}. \tag{4}$$

The counting distribution for zero photons and one photon to be absorbed by the detector immersed in a one-photon field is given by:

$$P(0, t) = \mathrm{e}^{-\alpha_0 t} \qquad P(1, t) = 1 - \mathrm{e}^{-\alpha_0 t} \tag{5}$$

where $\alpha_0 = sN$.

3. Photocounting distribution

To obtain the counting distribution for an n-photon field, we use the two following results: (i) the independence of the atoms inside the detector (Rocca 1973, Arnedo and Rocca 1974); and (ii) the counting distribution for a one-photon field given in equation (5).

Consider an n-photon field. The probability of counting m photons is the probability that an event occurs m times in n independent trials. A counting experiment is a set of n successive counting experiments, each with a one-photon field. The probability is that of obtaining m heads in n tossings of a coin, which leads to the Bernouilli distribution

$$P(m, t) = \binom{n}{m} P(1, t)^m P(0, t)^{n-m} \tag{6}$$

where $P(1, t)$ and $P(0, t)$ are given in equation (5).

This result was obtained by Mollow (1968) and Scully and Lamb (1969), but their proof is more complicated than this one.

For a general field we weight the expression in equation (6), relative to the n-photon field, by the elements of the density matrix ρ_{nn}. The result is, in the mono-mode case

$$P(m, t) = \sum_n \rho_{nn} \binom{n}{m} \exp[-\alpha_0 t(n-m)][1-\exp(-\alpha_0 t)]^m \tag{7}$$

or

$$P(m, t) = \langle : \mathrm{e}^{-a^+a(1-\exp-\alpha_0 t)} (a^+a)^n [1-\exp(-\alpha_0 t)]^n / n! : \rangle. \tag{8}$$

For a classical field having a random intensity, the counting distribution is

$$P(m, t) = \langle \mathrm{e}^{-\gamma(t)} \gamma(t)^m / m! \rangle \tag{9}$$

where the parameter is proportional to a reduced intensity

$$\gamma(t) = \alpha_0 \int_0^t I(\theta)\, \mathrm{e}^{-\alpha_0 \theta}\, \mathrm{d}\theta. \tag{10}$$

From this new expression of the counting distribution (equations (8) and (9)), it is clear that intensity and time do not play the same role in a counting experiment. This observation was just made by Picinbono and Rousseau (1977).

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Appendix

Consider an atom immersed in a field. We begin with the semi-classical theory of disintegration of a discrete state coupled to a continuum given in Cohen-Tannoudji *et al* (1973). We shall use their notation for the atomic part. The results of that book can be generalized to quantized fields described by an operator $E(r, t)$ (Glauber 1963, 1964). The interaction Hamiltonian is

$$V(t) = -e \sum_\gamma q_\gamma(t) E(r, t). \tag{A.1}$$

At time t the system can be decomposed on the basis $\{|g \otimes \mathrm{i_F}\rangle, |a \otimes \mathrm{f_F}(K)\rangle\}$ as

$$\begin{aligned}\phi(t) = {} & b(g, t) \exp[-\mathrm{i}(E_g + E_{\mathrm{i_F}})t/\hbar] |g \otimes \mathrm{i_F}\rangle \\ & + \sum_K \int_0^\infty \mathrm{d}a\, b_K(a, t) \exp[-\mathrm{i}(E_a + E_{\mathrm{f_F}(K)})t/\hbar] |a \otimes \mathrm{f_F}(K)\rangle\end{aligned} \tag{A.2}$$

where $|g\rangle$ and $|a\rangle$ denote respectively the ground state and one of the excited states in the continuum of the atom; $|i_F\rangle$ and $|f_F(K)\rangle$ are the initial and final states of the electric field.

When the Schrödinger equation is applied to the state vector in equation (A.2), we obtain

$$i\hbar\frac{db(g,t)}{dt}=\sum_K\int_0^\infty da\ \exp\left[i\left(\frac{E_a-E_g}{\hbar}-\omega_K\right)t\right]\langle g|q|a\rangle\langle i_F|E(t)|f_F(K)\rangle b_K(a,t) \tag{A.3}$$

$$i\hbar\frac{db_K(a,t)}{dt}=\exp\left[-\left(\frac{E_a-E_g}{\hbar}-\omega_K\right)t\right]\langle a|q|g\rangle\langle f_F(K)|E^+(t)|i_F\rangle b_g(t) \tag{A.4}$$

where $\omega_K=(E_{i_F}-E_{f_F(K)})/\hbar$. The atom is supposed to be in its ground state at time $t=0$, therefore $b_K(a,t)$ depends only on $b(g,t)$. We obtain

$$\dot{b}(g,t)=-b(g,t)\sum_K\left(\frac{s_K}{2}+\frac{\delta\epsilon_K}{\hbar}\right)\langle i_F|E^+(t)|f_F(K)\rangle\langle f_F(K)|E^-(t)|i_F\rangle \tag{A.5}$$

where we have introduced the spectral response of the atom. This spectral response is generally broad compared to the spectrum of the incident light (Glauber 1964, Rousseau 1975). Let us consider the atomic spectral response defined by Cohen-Tannoudji *et al* (1973)

$$K(E_a)=\int d\beta|\langle\beta,E_a|q|g\rangle|^2\xi(\beta,E_a). \tag{A.6}$$

In equation (A.5), we have set

$$s_K=\frac{2\pi}{h}K(E_g+\hbar\omega_K)\qquad \delta\epsilon_K=\int_0^\infty\frac{K(E_a)}{E_g+\hbar\omega_K-E_a}dE_a. \tag{A.7}$$

The coefficients in equation (A.7) are independent of the mode K. Using the completeness relation for the states $|f_F(K)\rangle$, we obtain the simple differential equation for $b(g,t)$

$$\dot{b}(g,t)=-b(g,t)(s+i\delta\epsilon)n/\hbar \tag{A.8}$$

where n is the total number of photons ($n=\Sigma_K n_K$) of the field in the Fock state $|i_F\rangle=|\{n_K\}|$. The probability that the atom remain in the ground state up to time t is $|b(g,t)|^2$, and the transition probability for a $\{n_K\}$ photon field is

$$P^{(1)}(t)=1-e^{-stn}. \tag{A.9}$$

In the case of a general stationary field whose density matrix has diagonal elements on the Fock basis, or has a P representation, the transition probability becomes

$$P^{(1)}(t)=1-\sum_K\rho_{\{n_K,n_K\}}e^{-st\Sigma n_K} \tag{A.10}$$

or

$$P^{(1)}(t)=1-\int P\{\alpha_K\}\prod_K\exp[-|\alpha_K|^2(1-e^{-st})]\,d^2\alpha_K. \tag{A.11}$$

Alternatively this transition probability can be written in terms of the classical undimensional intensity I of the field

$$P^{(1)}(t)=1-\langle\exp[-I(1-e^{-st})]\rangle,$$

for a mono-mode field, or

$$P^{(1)}(t) = 1 - \left\langle \exp\left(-s \int_0^t I(\theta)\,\mathrm{d}\theta\right)\right\rangle \tag{A.12}$$

for a multi-mode field.

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